Robotics
Measures and errors

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http://robot.unipv.it/toolleeo

Measures and their processing

the reasons that lead to the continuous refinement of sensor technology are related to 2 important human activities:

1. the measurement of physical quantities
2. the processing of measured values

measurement and processing are tasks tightly related with the civilization of the human being

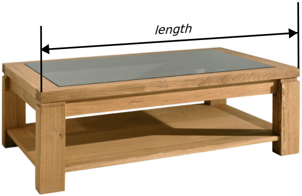
The measurement

Definition

measurement is the process that assignes numbers to entities or events of the real world

a measured value or an event is mapped to a range of values

The measurement



the claim
"*The length of the table is* 180 *cm*"
is meaningless from the scientific viewpoint

claims that are meaningful from the scientific perspective are

- "*The length of the table is between* 180 *and* 181 *cm*"
- "*The length of the table is* 180.5 *cm with an error of* ± 0.5 *cm*"

Attributes of a measure

> "*the length of the table is 180.5 cm*
> *with an error of* ± 0.5 *cm*"

the following attributes must be associated to a measure:

1.	the value (a number)	180.5
2.	the measurement unit	cm
3.	a specifier	length
4.	the origin	the table
5.	the error	± 0.5 cm

- all attributes contribute to define the measure
- in a computing system (e.g. a control apparatus) everything but the value can be neglected

Representation of a measure

a measure, together with the error, can be represented as follows:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

it means that there is a reasonable degree of certainty
that the measured value falls in the range
$[x^* - \delta_x, x^* + \delta_x]$

- the value x^* represents the best available approximation of the measured value
- δ_x is called absolute error

recalling the previous example:

$$x^* = 180.5 \qquad \delta_x = 0.5$$

The discrepancy

the same physical phenomenon or condition can bring to different measured values (including the error)

> the discrepancy is the difference between measured values of the same quantity or phenomenon

the discrepancy can be

1. **significant**: error ranges do not overlap
2. **not significant**: error ranges are overlapping

The "true value"

the **true value** is the value associated with a perfectly defined quantity, measured under the conditions of definition

some observations:

1. it would indicate the measured value if it were possible to get a perfect fit
2. the quantum mechanics determines the impossibility to get a perfect fit (Heisenberg's uncertainty principle)
3. $\Rightarrow$ the true value is an abstraction

we thus consider conventional true value: a value close enough to the true value, such that it differs by an amount (still unknown) which is not significant for the use of the value

Relative error

- we have considered absolute errors so far ($x^* \pm \delta_x$)
- absolute errors are important, however...
- they may jeopardize the evaluation between values with different orders of magnitude

> to deal with errors affecting values on different orders of magnitude, the relative error is used

$$\text{(relative error)} = \frac{\delta_x}{|x^*|}$$

Relative error

example: absolute error $\delta_x = 2$ cm

1. it has a given impact if $x^* = 180$ cm
2. the impact is much higher if $x^* = 10$ cm

$$\text{(relative error)} = \frac{\delta_x}{|x^*|}$$

considering the previous example, the relative error is:

1. $2/180 = 0.0111 = 1.11\%$
2. $2/10 = 0.2 = 20\%$

Absolute vs relative errors

the two following representations of errors are equivalent:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

$$(\text{measured value of } x) = x^* \left(1 \pm \frac{\delta_x}{|x^*|}\right)$$

Propagation of errors

measured values are typically used to:

- compute other values
- compare values

some questions arise:

- what is the effect of measurement errors on computed values?
- what is the role played in the comparison between values?

Substraction between values

known values:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

$$(\text{measured value of } y) = y^* \pm \delta_y$$

desired value to compute:

$$q = x - y$$

can be expressed as:

$$(\text{computed value of } q) = q^* \pm \delta_q$$

Substraction between measured values

the best approximation of the measured value is

$$q^* = x^* - y^*$$

since x^* and y^* are the best available approximations of measured values

Substraction between measured values

the error δ_q is obtained considering the highest and lowest possible values of $(x - y)$

- the highest value corresponds to $x = x^* + \delta_x$ and $y = y^* - \delta_y$
- the lowest value corresponds to $x = x^* - \delta_x$ and $y = y^* + \delta_y$

the highest possible value is

$$\max(x - y) = (x^* + \delta_x) - (y^* - \delta_y) = x^* - y^* + (\delta_x + \delta_y)$$

the lowest possible value is

$$\min(x - y) = (x^* - \delta_x) - (y^* + \delta_y) = x^* - y^* - (\delta_x + \delta_y)$$

therefore

$$\delta_q = (\delta_x + \delta_y)$$

Substraction between measured values

summarizing

$$q = x - y = q^* \pm \delta_q$$

where

$$q^* = x^* - y^*$$

$$\delta_q = \delta_x + \delta_y$$

Multiplication between measured values

known values:

$$(\text{measured value of } x) = x^* \left(1 \pm \frac{\delta_x}{|x^*|}\right)$$

$$(\text{measured value of } y) = y^* \left(1 \pm \frac{\delta_y}{|y^*|}\right)$$

the desired outcome is:

$$q = x \cdot y$$

that can be expressed as:

$$(\text{computed value of } q) = q^* \left(1 \pm \frac{\delta_q}{|q^*|}\right)$$

Multiplication between measured values

the best possible approximation of the calculated value is

$$q^* = x^* \cdot y^*$$

since x^* and y^* are the best available approximations of measured values

Multiplication between measured values

the error δ_q can be obtained considering the highest and lowest values of $x \cdot y$

- the highest value corresponds to $x = x^*(1 + \delta_x/|x^*|)$ and $y = y^*(1 + \delta_y/|y^*|)$
- the lowest value corresponds to $x = x^*(1 - \delta_x/|x^*|)$ and $y = y^*(1 - \delta_y/|y^*|)$

the highest possible value is

$$\begin{aligned}\max(x \cdot y) &= x^* y^* \left(1 + \frac{\delta_x}{|x^*|}\right)\left(1 + \frac{\delta_y}{|y^*|}\right) \\ &= x^* y^* \left(1 + \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|} + \frac{\delta_x}{|x^*|}\frac{\delta_y}{|y^*|}\right)\end{aligned}$$

Multiplication between measured values

when $\frac{\delta_x}{|x^*|} \ll 1$ and $\frac{\delta_y}{|y^*|} \ll 1$
we can neglect the product of relative errors

the result is

$$\max(x \cdot y) = x^* y^* \left(1 + \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|} + \cancel{\frac{\delta_x}{|x^*|}\frac{\delta_y}{|y^*|}}\right)$$

the same procedure can be repeated for the lowest value $\min(x \cdot y)$

resulting error:

$$\frac{\delta_q}{|q^*|} = \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|}$$

Multiplication between measured values

summarizing:

$$q = x \cdot y = q^* \pm \delta_q$$

with

$$q^* = x^* \cdot y^*$$

$$\frac{\delta_q}{|q^*|} = \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|}$$

Summary

in case of substraction between measured values:

the absolute error on the result is
equal to the sum of absolute errors on measured values

in case of multiplication between measured values:

the relative error on the result is
equal to the sum of relative errors of measured values

Multiplication by a constant number

known values:

$$\text{(measured value of } x) = x^* \pm \delta_x$$

$$\text{known value } A$$

the desired value is:

$$q = Ax$$

the error is

$$\delta_q = |A|\delta_x$$

- it is a multiplication: relative errors sum up
- the error on A is null
- in the formula $\frac{\delta_q}{|q|} = \frac{\delta_x}{|x|}$ it suffices to assign $|q| = |Ax|$

Sum and division

with the same procedure adopted for substractions and multiplications, it can be shown that the same results hold for sums and divisions

in case of sum of measured values

> the absolute error on the result is
> equal to the sum of absolute errors on measured values

in case of division of measured values

> the relative error on the result is
> equal to the sum of relative errors of measured values

Function of a variable

known values:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

the desired value is:

$$q = f(x)$$

that can be expressed as:

$$(\text{computed value of } q) = q^* \pm \delta_q$$

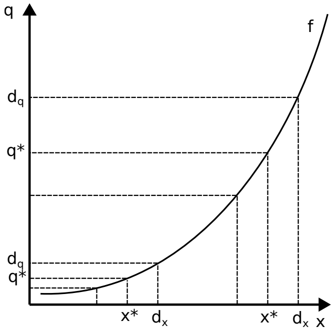
Function of a variable

the best approximation of the value to be computed is

$$q^* = f(x^*)$$

while the error is

$$\delta_q = \left| \frac{df}{dx}(x^*) \right| \delta_x$$



Function of many variables

known values:

$$\text{measured values } x_1, \ldots, x_n$$

$$(\text{measured value } x_i) = x_i^* \pm \delta_{x_i}$$

the desired value is:

$$q = f(x_1, \ldots, x_n)$$

that can be expressed as:

$$(\text{computed value of } q) = q^* \pm \delta_q$$

Function of many variables

the best approximation of the computed value is

$$q^* = f(x_1^*, \ldots, x_n^*)$$

while the error is

$$\delta_q = \left|\frac{\partial f}{\partial x_1}(x_1^*)\right| \delta x_1 + \ldots + \left|\frac{\partial f}{\partial x_n}(x_n^*)\right| \delta x_n$$

Example of error propagation in complex expressions

x and y are two measured values with known errors, i.e.:

$$x = x^* \pm \delta_x \qquad \text{and} \qquad y = y^* \pm \delta_y$$

the desired values is

$$q = a \cdot x + b \cdot y + x \cdot y$$

where a and b are known and constant coefficients and

$$q = q^* \pm \delta_q$$

Example of error propagation in complex expressions

$$q = q^* \pm \delta_q = a \cdot x + b \cdot y + x \cdot y$$

the best approximation q^* is

$$q^* = a \cdot x^* + b \cdot y^* + x^* \cdot y^*$$

Example of error propagation in complex expressions

$$q = q^* \pm \delta_q = a \cdot x + b \cdot y + x \cdot y$$

to calculate δ_q the following steps are applied:

1. abs $\mathrm{err}(a \cdot x) = \delta_{(a \cdot x)} = a \cdot \delta_x$
2. abs $\mathrm{err}(b \cdot y) = \delta_{(b \cdot y)} = b \cdot \delta_y$
3. abs $\mathrm{err}(x \cdot y) = \delta_{(x \cdot y)}$:
   - rel $\mathrm{err}(x) = \delta_x / x^*$
   - rel $\mathrm{err}(y) = \delta_y / y^*$
   - rel $\mathrm{err}(x \cdot y) =$ rel $\mathrm{err}(x) +$ rel $\mathrm{err}(y)$
   - abs $\mathrm{err}(x \cdot y) =$ rel $\mathrm{err}(x \cdot y) \cdot (x^* \cdot y^*)$
4. abs $\mathrm{err}(q) = \delta_q = \delta_{(a \cdot x)} + \delta_{(b \cdot y)} + \delta_{(x \cdot y)}$

Precision

the precision is the degree of convergence of data individually collected on an average value of the series to which they belong

- the dispersion of values can be produced by non-repeatable random variations (statistical error)
- to obtain a reliable average value it is necessary to make a sufficiently large number of observations
- in statistic, precision is expressed in terms of standard deviation

Precision

the precision has the following features:

- repeatability: the variation due to the measuring instrument, is the dispersion of values obtained using the same tools, by the same operator, under the same conditions and in a reasonably short time
- reproducibility: the variation due to the system to be measured; it is the dispersion due to measure the same quantity, unsing different instruments and/or by different operators, and/or on a relatively long time

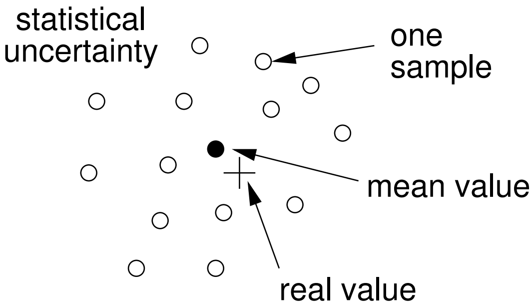
Precision

> the precision is a statistical characteristic of measurements

although someone does not have a good opinion of statistics:

- 94.5% of statistics are wrong (Woody Allen)
- the futility of statistics is statistically demonstrated (Umberto Domina)
- the statistician is a man who makes the right calculation starting from dubious premises to get to a wrong result (Jean Delacour)
- torture the data long enough and they confess whatever (Gregg Easterbrook)

Statistical error



Statistical error

deviation between the measured values and its mean value

Statistical error

given the following statistical values:

mean value

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

standard deviation:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

the statistical error is usually expressed as a ratio between the standard deviation and the mean value, as a percentage

$$err = \sigma / |\overline{x}|$$

Statistical and standard error

given a set of measurements of the same quantity, it holds:

- the mean value represents the best approximation of the measured quantity
- uncertainty is related to the standard deviation, which, in the case of normal distribution (typical), ensures that 68% of measures fall in the range

$$\overline{x} \pm \sigma$$

the standard error is defined to express that the more measurements are made and the more the estimate of the uncertainty improves:

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$$

Accuracy

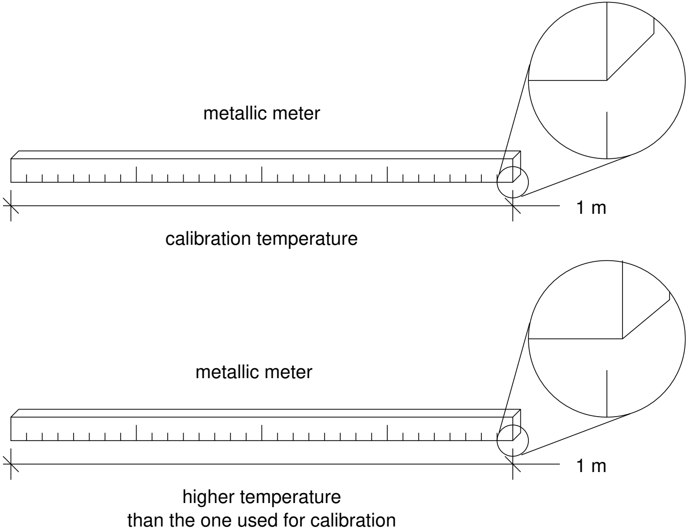
the accuracy is the degree of correspondence of the theoretical data, determined from a series of measured values (e.g. the average value of several measurements), with the true value or reference

> the constant and repeatable error that is obtained is the systematic error (or bias)

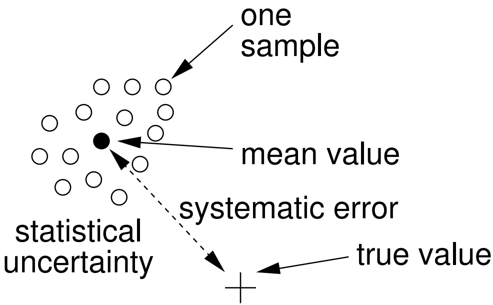
the accuracy can be characterized by three components:

- linearity: it considers the effect of the measurement range on the accuracy of the measurement itself
- accuracy (actually): it is the difference between the average of the measured values and a reference sample
- stability: the accuracy of the measurement over time; it considers the variation in time of the measurement of the same instrument, on the same sample

Example


metallic meter
1 m
calibration temperature
metallic meter
1 m
higher temperature
than the one used for calibration

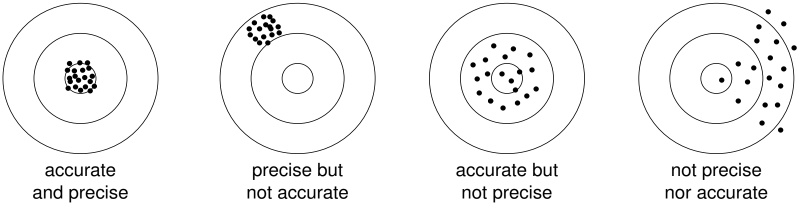
Systematic error



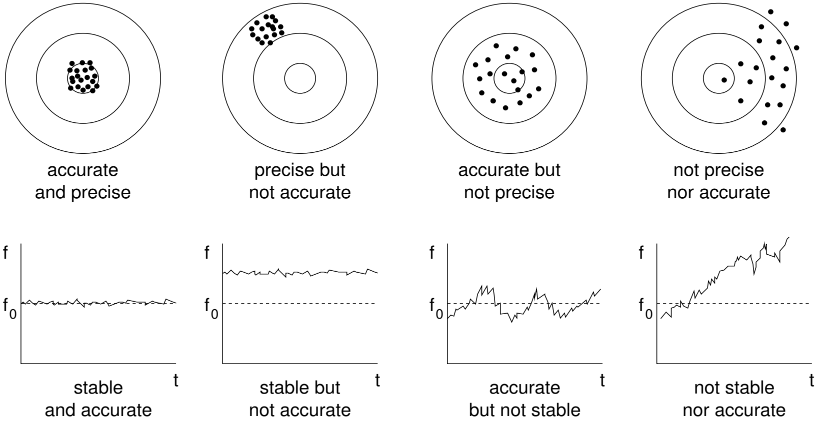
Systematic error

deviation between the mean value and the true value

Stability and accuracy of a signal



Stability and accuracy of a signal



Error, accuracy and precision

- the statistical error is relatively easy to evaluate, since it only requires to calculate the standard deviation of the distribution of the measured values
- the systematic error is more complex; in general, it is due to calibration errors or changes in parameters of the measuring instrument due, e.g., to the temperature
- a tool which is deteriorated or altered, used to acquire a set of values, can be precise, since the obtained measures are close to each other, but can be poorly accurate if these values differ significantly from the true value

Calibration of a sensor

suppose we want to calibrate an instrument for measuring a distance

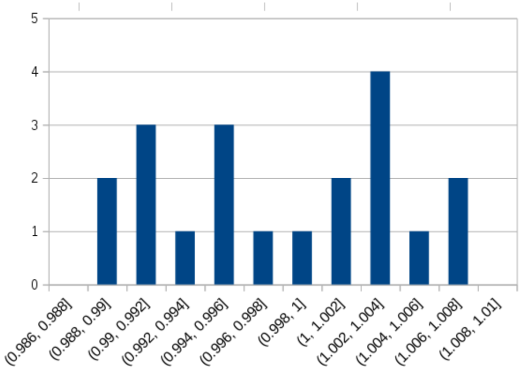
the reference sample is 1 m

#	measure [m]	#	measure [m]
1	0.990	11	0.995
2	1.007	12	1.004
3	1.004	13	1.003
4	0.991	14	1.000
5	0.989	15	0.992
6	1.008	16	0.994
7	0.997	17	1.005
8	1.002	18	0.995
9	0.996	19	0.991
10	1.001	20	1.004

how can the accuracy of an instrument be evaluated?

Calibration of a sensor

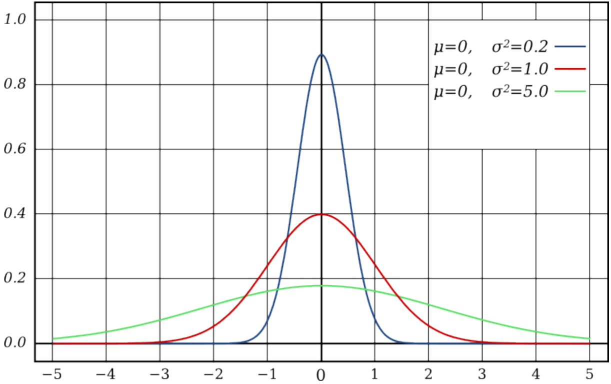
- mean value $\overline{x} = 0.9984$ m (-1.6 mm)
- standard deviation $s = 0.0061$ (6.1 mm)



frequencies in the range (0.986, 1.010] m
each bar refers to a range size $\Delta = 0.002$m

Calibration of a sensor and limit distribution

if it is possible to perform an infinite number of measurements and setting $\Delta \to 0$, the histogram (usually) tends to get the shape of the limit distribution



Normal distribution

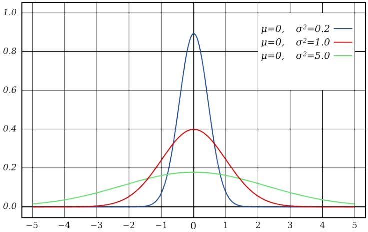
the function that defines the bell shape is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

it is said "Gaussian function"

the distribution of measured values associated with the Gaussian function is said normal distribution

Sensor calibration and normal distribution



if a normal distribution is assumed, then it holds that

- 68% of sampled values are in the range $\mu \pm \sigma$ ($\overline{x} \pm s$)
- 95% of sampled values are in the range $\mu \pm 2\sigma$ ($\overline{x} \pm 2s$)
- 99.7% of sampled values are in the range $\mu \pm 3\sigma$ ($\overline{x} \pm 3s$)