Robotics
Measures and errors

Tullio Facchinetti
<tullio.facchinetti@unipv.it>

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http://robot.unipv.it/toolleeo

Measures and their processing

the reasons that lead to the continuous refinement of sensor technology are related to 2 important human activities:

1. the measurement of physical quantities
2. the processing of measured values

measurement and processing are tasks tightly related with the civilization of the human being

Physical quantities and their measurement

- appropriate procedures are established to compare two values of the same class and by means of these processes it is defined when the two values are equal and when they are unequal, and in this second case which is the greater and which the lesser
- it is possible to establish, in a simple infinity of different ways, a bijective correspondence, ordered and reciprocal, between all variables of the same class and all positive real numbers; these bijective correspondences establish that the same value is associated with the same numbers and viceversa, while different numbers are associated with different values and viceversa

Euclid, "Book V of the Elements" (IV-III century BC)

Physical quantities and their measurement

"is called a measurement of a quantity, in the most general sense, any method by which a unique and reciprocal correspondence between all or certain variables of a certain kind and all or some integers is established, rational or real according to the case. In this general sense, the measurement requires a one-to-one relationship between the numbers and the values of the considered quantity; the relationship can be direct or indirect"

Bertrand Russel, "Principia Mathematica" (1910-1913)

Applications of measurement

applications that are affected by measurement processes can be classified in one of the following types (or a combination thereof):

1. monitoring
2. control

- **monitoring:** refers to the simple measurement of system parameters, to assess its progress or behavior
- **control:** refers to the use of measurements to determine the actions to be implemented on a system to adjust its operation (e.g. feedback loops)

The measurement

Definition

measurement is the process that assignes numbers to entities or events of the real world

a measured value or an event is mapped to a range of values

The measurement

the claim
"*The height of Sebastian is* 70 *cm*"
is meaningless from the scientific viewpoint

claims that are meaningful from the scientific perspective are

- "*The height of Sebastian is between* 70 *and* 71 *cm*"
- "*The height of Sebastian is* 70.5 *cm with an error of* ± 0.5 *cm*"

Attributes of a measure

> "*the height of Sebastian is 70.5 cm*
> *with an error of* ± 0.5 *cm*"

the following attributes must be associated to a measure:

1.	the value (a number)	70.5
2.	the measurement unit	cm
3.	a specifier	height
4.	the origin	Sebastian
5.	the error	± 0.5 cm

- all attributes contribute to define the measure
- in a computing system (e.g. a control apparatus) everything but the value can be neglected

Representation of a measure

a measure, together with the error, can be represented as follows:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

it means that there is a reasonable degree of certainty
that the measured value falls in the range
$[x^* - \delta_x, x^* + \delta_x]$

- the value x^* represents the best available approximation of the measured value
- δ_x is called absolute error

recalling the previous example:

$$x^* = 70.5 \qquad \delta_x = 0.5$$

The discrepancy

the same physical phenomenon or condition can bring to different measured values (including the error)

> the discrepancy is the difference between measured values of the same quantity or phenomenon

the discrepancy can be

1. **significant**: error ranges do not overlap
2. **not significant**: error ranges are overlapping

The "true value"

the **true value** is the value associated with a perfectly defined quantity, measured under the conditions of definition

some observations:

1. it would indicate the measured value if it were possible to get a perfect fit
2. the quantum mechanics determines the impossibility to get a perfect fit (Heisenberg's uncertainty principle)
3. $\Rightarrow$ the true value is an abstraction

we thus consider conventional true value: a value close enough to the true value, such that it differs by an amount (still unknown) which is not significant for the use of the value

Relative error

- we have considered absolute errors so far ($x^* \pm \delta_x$)
- absolute errors are important, however...
- they may jeopardize the evaluation between values with different orders of magnitude

> to deal with errors affecting values on different orders of magnitude, the relative error is used

$$\text{(relative error)} = \frac{\delta_x}{|x^*|}$$

Relative error

example: absolute error $\delta_x = 2$ cm

1. it has a given impact if $x^* = 70$ cm
2. the impact is much higher if $x^* = 5$ cm

$$\text{(relative error)} = \frac{\delta_x}{|x^*|}$$

considering the previous example, the relative error is:

1. $2/70 = 0.0286 = 2.9\%$
2. $2/5 = 0.4 = 40\%$

Absolute vs relative errors

the two following representations of errors are equivalent:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

$$(\text{measured value of } x) = x^* \left(1 \pm \frac{\delta_x}{|x^*|}\right)$$

Propagation of errors

measured values are typically used to:

- compute other values
- compare values

some questions arise:

- what is the effect of measurement errors on computed values?
- what is the role played in the comparison between values?

Substraction between values

known values:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

$$(\text{measured value of } y) = y^* \pm \delta_y$$

desired value to compute:

$$q = x - y$$

can be expressed as:

$$(\text{computed value of } q) = q^* \pm \delta_q$$

Substraction between measured values

the best approximation of the measured value is

$$q^* = x^* - y^*$$

since x^* and y^* are the best available approximations of measured values

Substraction between measured values

the error δ_q is obtained considering the highest and lowest possible values of $(x - y)$

- the highest value corresponds to $x = x^* + \delta_x$ and $y = y^* - \delta_y$
- the lowest value corresponds to $x = x^* - \delta_x$ and $y = y^* + \delta_y$

the highest possible value is

$$\max(x - y) = (x^* + \delta_x) - (y^* - \delta_y) = x^* - y^* + (\delta_x + \delta_y)$$

the lowest possible value is

$$\min(x - y) = (x^* - \delta_x) - (y^* + \delta_y) = x^* - y^* - (\delta_x + \delta_y)$$

therefore

$$\delta_q = (\delta_x + \delta_y)$$

Substraction between measured values

summarizing

$$q = x - y = q^* \pm \delta_q$$

where

$$q^* = x^* - y^*$$

$$\delta_q = \delta_x + \delta_y$$

Multiplication between measured values

known values:

$$(\text{measured value of } x) = x^* \left(1 \pm \frac{\delta_x}{|x^*|}\right)$$

$$(\text{measured value of } y) = y^* \left(1 \pm \frac{\delta_y}{|y^*|}\right)$$

the desired outcome is:

$$q = x \cdot y$$

that can be expressed as:

$$(\text{computed value of } q) = q^* \left(1 \pm \frac{\delta_q}{|q^*|}\right)$$

Multiplication between measured values

the best possible approximation of the calculated value is

$$q^* = x^* \cdot y^*$$

since x^* and y^* are the best available approximations of measured values

Multiplication between measured values

the error δ_q can be obtained considering the highest and lowest values of $x \cdot y$

- the highest value corresponds to $x = x^*(1 + \delta_x/|x^*|)$ and $y = y^*(1 + \delta_y/|y^*|)$
- the lowest value corresponds to $x = x^*(1 - \delta_x/|x^*|)$ and $y = y^*(1 - \delta_y/|y^*|)$

the highest possible value is

$$\begin{aligned}\max(x \cdot y) &= x^* y^* \left(1 + \frac{\delta_x}{|x^*|}\right)\left(1 + \frac{\delta_y}{|y^*|}\right) \\ &= x^* y^* \left(1 + \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|} + \frac{\delta_x}{|x^*|}\frac{\delta_y}{|y^*|}\right)\end{aligned}$$

Multiplication between measured values

when $\frac{\delta_x}{|x^*|} \ll 1$ and $\frac{\delta_y}{|y^*|} \ll 1$
we can neglect the product of relative errors

the result is

$$\max(x \cdot y) = x^* y^* \left(1 + \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|} + \cancel{\frac{\delta_x}{|x^*|}\frac{\delta_y}{|y^*|}}\right)$$

the same procedure can be repeated for the lowest value $\min(x \cdot y)$

resulting error:

$$\frac{\delta_q}{|q^*|} = \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|}$$

Multiplication between measured values

summarizing:

$$q = x \cdot y = q^* \pm \delta_q$$

with

$$q^* = x^* \cdot y^*$$

$$\frac{\delta_q}{|q^*|} = \frac{\delta_x}{|x^*|} + \frac{\delta_y}{|y^*|}$$

Summary

in case of substraction between measured values:

> the absolute error on the result is
> equal to the sum of absolute errors on measured values

in case of multiplication between measured values:

> the relative error on the result is
> equal to the sum of relative errors of measured values

Multiplication by a constant number

known values:

$$(\text{measured value of } x) = x^* \pm \delta_x$$

$$\text{known value } A$$

the desired value is:

$$q = Ax$$

the error is

$$\delta_q = |A|\delta_x$$

- it is a multiplication: relative errors sum up
- the error on A is null
- in the formula $\frac{\delta_q}{|q|} = \frac{\delta_x}{|x|}$ it suffices to assign $|q| = |Ax|$

Sum and division

with the same procedure adopted for substractions and multiplications, it can be shown that the same results hold for sums and divisions

in case of sum of measured values

> the absolute error on the result is
> equal to the sum of absolute errors on measured values

in case of division of measured values

> the relative error on the result is
> equal to the sum of relative errors of measured values

Function of a variable

known values:

$$\text{(measured value of } x) = x^* \pm \delta_x$$

the desired value is:

$$q = f(x)$$

that can be expressed as:

$$\text{(computed value of } q) = q^* \pm \delta_q$$

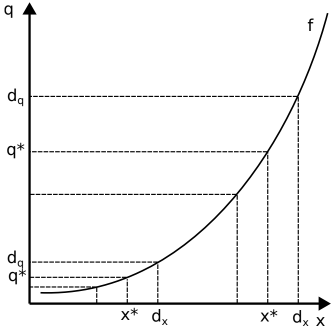
Function of a variable

the best approximation of the value to be computed is

$$q^* = f(x^*)$$

while the error is

$$\delta_q = \left| \frac{df}{dx}(x^*) \right| \delta_x$$



Function of many variables

known values:

$$\text{measured values } x_1, \ldots, x_n$$

$$(\text{measured value } x_i) = x_i^* \pm \delta_{x_i}$$

the desired value is:

$$q = f(x_1, \ldots, x_n)$$

that can be expressed as:

$$(\text{computed value of } q) = q^* \pm \delta_q$$

Function of many variables

the best approximation of the computed value is

$$q^* = f(x_1^*, \ldots, x_n^*)$$

while the error is

$$\delta_q = \left|\frac{\partial f}{\partial x_1}(x_1^*)\right| \delta x_1 + \ldots + \left|\frac{\partial f}{\partial x_n}(x_n^*)\right| \delta x_n$$

Example of error propagation in complex expressions

- being x and y two measured values with known errors

the desired values is

$$\sqrt{a \cdot x^2 - b \cdot y^2}$$

where a and b are known and constant coefficients

the steps to calculate the error on the final result are

- calculate the error on x^2 (or $x \cdot x$)
- multiply the calculated error by a
- calculate the error on y^2 (or $y \cdot y$)
- multiply the calculated error by b
- calculate the error on the difference between the two previous computed values
- calculate the error on the square root of this latest computed value

Sources of errors

Where do errors come from in measurements?

there is a number of sources of errors in measurements...

Sources of errors

1

imprecise definition of
the system/process/entity to measure

examples:

- "percentage of potassium in the Adriatic Sea": the entity to measure is not completely defined; the measurement may depend on the location where the sample is taken
- "the gravity acceleration at the sea level": the variable to measure also depends on the latitude

Sources of errors

2

the system/process/entity to measure is
hard/impossible to "isolate" or access

examples:

- "the average decay time of an isotope X"
- "the acceleration of an object over an inclined plane *without friction*"
- "the temperature of a star"

Sources of errors

3

the reference sample does not represent the system/process/entity to measure

examples:

- "the percentage of people that is taller than 1.80 cm": due to the lack of financial resources or available time, an exaustive sampling might not be possible; an estimation is thus done on a subset of persons; the selection of the subset introduces the approximation

Sources of errors

4

the reference sample
is altered w.r.t. expected conditions

examples:

- the reference sample or the measuring instrument is damaged

Sources of errors

5

influence of environmental conditions

examples:

- the temperature during the measurement significantly differs from the reference conditions of the instrument

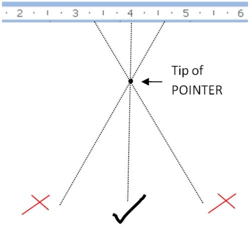
Sources of errors

6

error while reading the measurement tool

examples:

- e.g., due to parallax



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Sources of errors

7

limited resolution of the measurement tool

examples:

- the measured value changes significantly but the variation is bounded within the resolution limit of the measurement tool, which therefore does not register any variation

Sources of errors

8

in repeated measurements, there can be variations on the measurement conditions that are not taken into account

examples:

- the same phenomenon/event/quantity is measured
- the same, well calibrated measurement gears are used
- there are changing environmental parameters like temperature, pressure, etc.

Precision

the precision is the degree of convergence of data individually collected on an average value of the series to which they belong

- the dispersion of values can be produced by non-repeatable random variations (statistical error)
- to obtain a reliable average value it is necessary to make a sufficiently large number of observations
- in statistic, precision is expressed in terms of standard deviation

Precision

the precision has the following features:

- repeatability: the variation due to the measuring instrument, is the dispersion of values obtained using the same tools, by the same operator, under the same conditions and in a reasonably short time
- reproducibility: the variation due to the system to be measured; it is the dispersion due to measure the same quantity, unsing different instruments and/or by different operators, and/or on a relatively long time

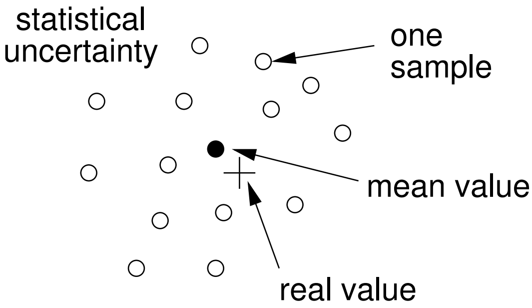
Precision

the precision is a statistical characteristic of measurements

although someone does not have a good opinion of statistics:

- 94.5% of statistics are wrong (Woody Allen)
- the futility of statistics is statistically demonstrated (Umberto Domina)
- the statistician is a man who makes the right calculation starting from dubious premises to get to a wrong result (Jean Delacour)
- torture the data long enough and they confess whatever (Gregg Easterbrook)

Statistical error



Statistical error

deviation between the measured values and its mean value

Statistical error

given the following statistical values:

mean value

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

standard deviation:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

the statistical error is usually expressed as a ratio between the standard deviation and the mean value, as a percentage

$$err = \sigma / |\overline{x}|$$

Statistical and standard error

given a set of measurements of the same quantity, it holds:

- the mean value represents the best approximation of the measured quantity
- uncertainty is related to the standard deviation, which, in the case of normal distribution (typical), ensures that 68% of measures fall in the range

$$\overline{x} \pm \sigma$$

the standard error is defined to express that the more measurements are made and the more the estimate of the uncertainty improves:

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$$

Accuracy

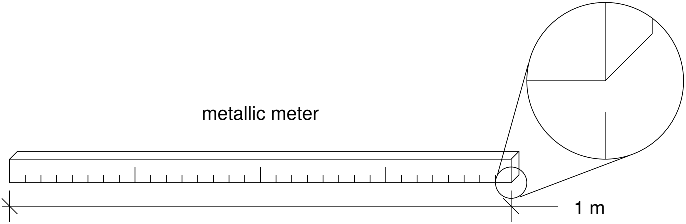
the accuracy is the degree of correspondence of the theoretical data, determined from a series of measured values (e.g. the average value of several measurements), with the true value or reference

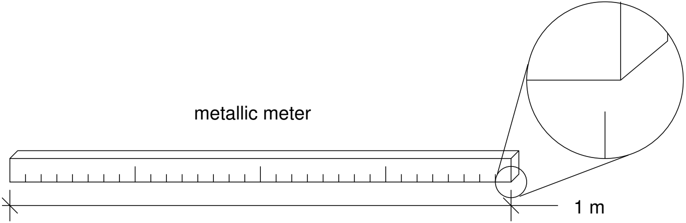
> the constant and repeatable error that is obtained is the systematic error (or bias)

the accuracy can be characterized by three components:

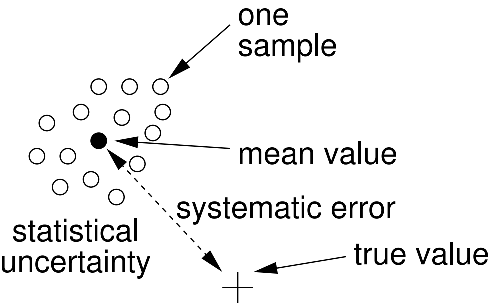
- linearity: it considers the effect of the measurement range on the accuracy of the measurement itself
- accuracy (actually): it is the difference between the average of the measured values and a reference sample
- stability: the accuracy of the measurement over time; it considers the variation in time of the measurement of the same instrument, on the same sample

Example


metallic meter
1 m
calibration temperature


metallic meter
1 m
higher temperature
than the one used for calibration

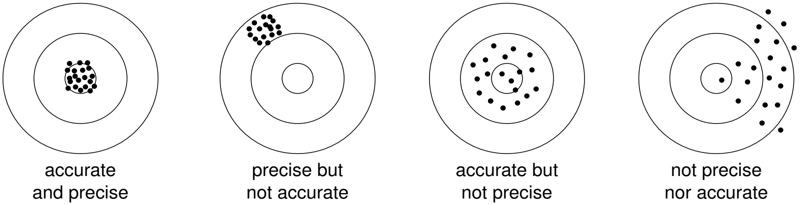
Systematic error



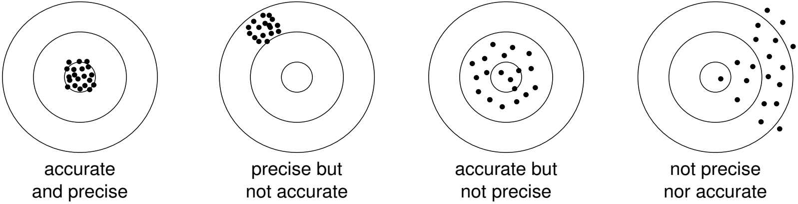
Systematic error

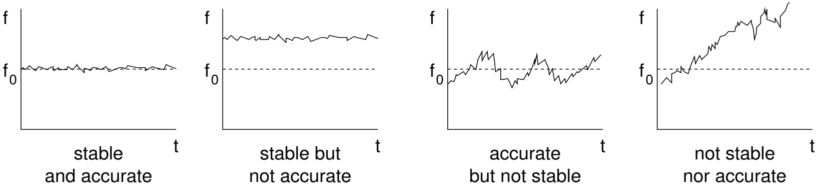
deviation between the mean value and the true value

Stability and accuracy of a signal



Stability and accuracy of a signal





Error, accuracy and precision

- the statistical error is relatively easy to evaluate, since it only requires to calculate the standard deviation of the distribution of the measured values
- the systematic error is more complex; in general, it is due to calibration errors or changes in parameters of the measuring instrument due, e.g., to the temperature
- a tool which is deteriorated or altered, used to acquire a set of values, can be precise, since the obtained measures are close to each other, but can be poorly accurate if these values differ significantly from the true value

Calibration of a sensor

suppose we want to calibrate an instrument for measuring a distance

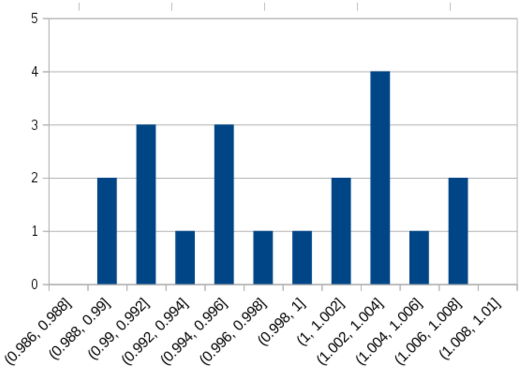
the reference sample is 1 m

#	measure [m]	#	measure [m]
1	0.990	11	0.995
2	1.007	12	1.004
3	1.004	13	1.003
4	0.991	14	1.000
5	0.989	15	0.992
6	1.008	16	0.994
7	0.997	17	1.005
8	1.002	18	0.995
9	0.996	19	0.991
10	1.001	20	1.004

how can the accuracy of an instrument be evaluated?

Calibration of a sensor

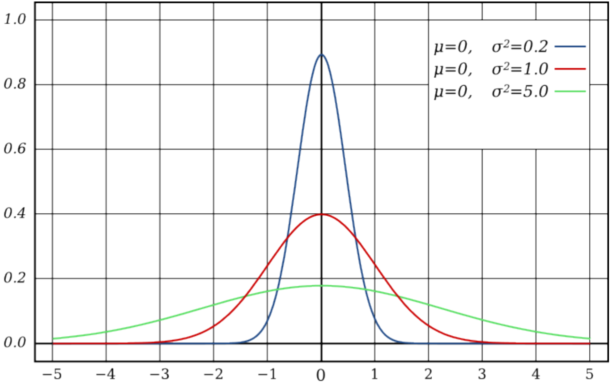
- mean value $\overline{x} = 0.9984$ m (-1.6 mm)
- standard deviation $s = 0.0061$ (6.1 mm)



frequencies in the range (0.986, 1.010] m
each bar refers to a range size $\Delta = 0.002$m

Calibration of a sensor and limit distribution

if it is possible to perform an infinite number of measurements and setting $\Delta \to 0$, the histogram (usually) tends to get the shape of the limit distribution



Normal distribution

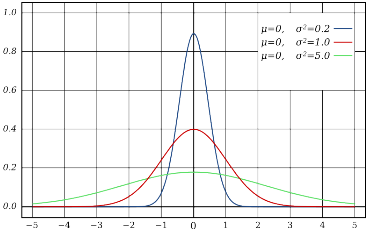
the function that defines the bell shape is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

it is said "Gaussian function"

the distribution of measured values associated with the Gaussian function is said normal distribution

Sensor calibration and normal distribution



if a normal distribution is assumed, then it holds that

- 68% of sampled values are in the range $\mu \pm \sigma$ ($\overline{x} \pm s$)
- 95% of sampled values are in the range $\mu \pm 2\sigma$ ($\overline{x} \pm 2s$)
- 99.7% of sampled values are in the range $\mu \pm 3\sigma$ ($\overline{x} \pm 3s$)