Robotics
Intelligent sensors (part 1)

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Tuesday 8th January, 2018

http://robot.unipv.it/toolleeo

position sensors allow the measurement of a position or a displacement

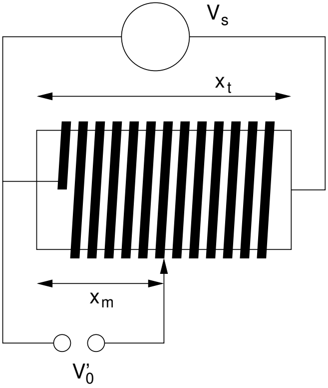
- positions can be **absolute** or **relative**
- relative positions are measured by displacement sensors

position sensors are building blocks of many other sensors, such as velocity, acceleration, force, and pressure sensors

there are sensors to measure translational (linear) and rotational positions

several different technologies are used to build a position sensor:

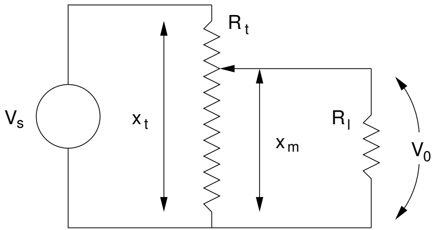
- resistive (potentiometers)
- capacitive
- ultrasound (proximity sensors)
- inductive (linear variable differential transformer)
- optical (proximity sensor, photodiode array, laser Doppler)
- Hall effect



the potentiometer is based on the principle of the variable resistance

$$V_0' = \frac{x_m}{x_t} V_s$$

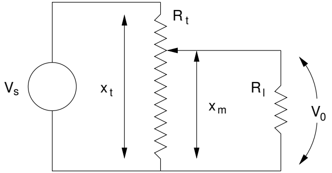
the relationship between V_0' and x_m is linear to measure a voltage a load needs to be connected to the output pins, and some non-null electric current must flow



$$\frac{V_0}{V_s} = \left(\frac{x_t}{x_m} + \frac{R_t}{R_l} \left(1 - \frac{x_m}{x_t} \right) \right)^{-1}$$

when R_l has a finite value, the relationship between V_0 and x_m is non-linear

The potentiometer: some considerations



$$\frac{V_0}{V_s} = \left(\frac{x_t}{x_m} + \frac{R_t}{R_l} \left(1 - \frac{x_m}{x_t} \right) \right)^{-1}$$

- for a given value of R_l, R_t must be reasonably low to reduce measurement errors
- the power voltage V_s can be increased to increase the sensitivity
- increasing the voltage leads to thermal dissipation issues, including melting

- the resolution is limited by the number of coils per unit of length
- manufacturing technologies pose limits to such a density
- a higher resolution can be obtained using carbon or ceramic films, metal films, or conductive plastic
- inductance can be high when used with alternate current

The potentiometer: some observations

the main problem of the potentiometer
arises at the contact point between
the sliding contact and the coil

- the contact wears, and it is sensible to environmental factors like humidity and dirt
- the potentiometer is guaranteed for a given number of full slidings (e.g., 10 millions)
- the lifetime strongly depends on the adopted material
- in case of quick movements, the contact may bounce, leading to intermittent signals

Potentiometer: example of datasheet

Precision Potentiometric Output
Ranges: 0-3 to 0-30 inches [0-75 to 0-750 mm]
3K – 10K ohms • IP65

CLWG

Specif cation Summary:

GENERAL

Full Stroke Ranges	0-3 to 0-30 in. (0-75 to 0-750 mm)
Output Signal	voltage divider (potentiometer)
Linearity	±0.04 to 0.1% full stroke, see ordercode
Repeatability	<0.01 mm
Resolution	essentially inf nite
Life Expectancy	50 million cycles
Enclosure Material	aluminum
Sensor	conductive plastic linear potentiometer
Operating Speed	200 inches (5 M) per second, max.

ELECTRICAL

Input Resistance	5K to 10K ohms (±20%), see ordercode
Recommended Maximum Input Voltage	25-30 V (AC or DC)
Recommended Operating Wiper Current	≤1 µA

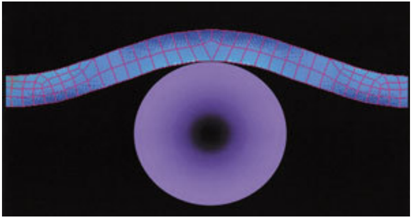
ENVIRONMENTAL

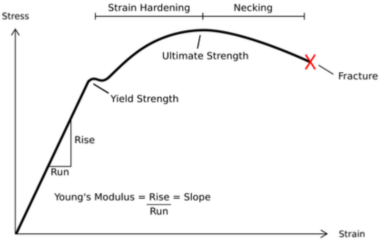
Enclosure Design	IP65
Operating Temperature	-22º to 212ºF
Vibration	up to 10 G's to 2000 Hz maximum

from Celesco Transducer Products Inc.

Strain gauge

- the strain (deformation) is a dimensionless geometrical measure of deformation due to the variation of length of a rigid mechanical body
- the strain be intended as the relative variation of the shape of one body





Hooke's law

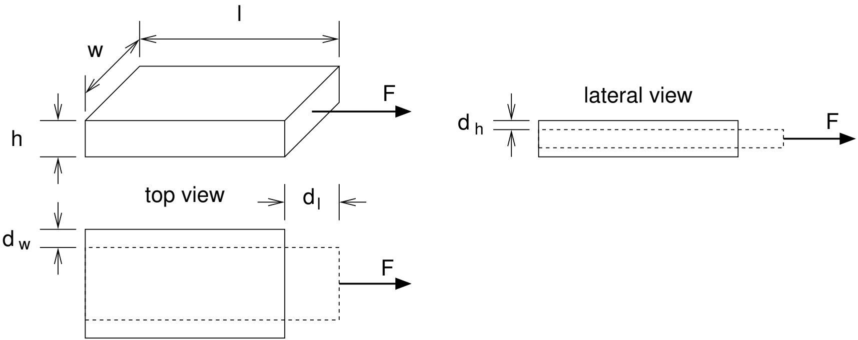
$$\sigma = \varepsilon E$$

it put into relationships:

- σ is the surface stress [pressure]
- ε is the relative length variation $\Delta l / l$ [dimensionless]
- E is the Young's modulus, a material property, [force/area]

the Hooke's law holds for
monodimensional strain gauges
made by **elastic material**
with **linear behavior** in the working range

Strain gauge



> a strain ε in one direction causes a deformation in the two orthogonal directions equal to $-\nu\varepsilon$

ν is called Poisson's ratio

Strain gauge: example

example of strain gauge that measures the deformation of a building

Strain and resistance

the resistance of a metal conductor having length l, cross-sectional area S and bulk resistivity ρ is

$$R = \rho \frac{l}{S}$$

applying a pressure σ $[N/m^2]$, the variation of resistance is

$$\begin{aligned} \frac{dR}{d\sigma} &= \frac{d}{d\sigma}\left(\rho \frac{l}{S}\right) \\ &= \frac{\rho}{S}\frac{\delta l}{\delta \rho} - \frac{\rho l}{S^2}\frac{\delta S}{\delta \rho} + \frac{l}{S}\frac{\delta \rho}{\delta \rho} \end{aligned}$$

dividing both members by the initial value of R leads to

$$\frac{1}{R}\frac{dR}{d\rho} = \frac{1}{l}\frac{\delta l}{\delta \rho} - \frac{1}{S}\frac{\delta S}{\delta \rho} + \frac{1}{\rho}\frac{\delta \rho}{\delta \rho}$$

by eliminating the ratio $d\rho$ from the following equation

$$\frac{1}{R}\frac{dR}{d\rho} = \frac{1}{l}\frac{\delta l}{\delta\rho} - \frac{1}{S}\frac{\delta S}{\delta\rho} + \frac{1}{\rho}\frac{\delta\rho}{\delta\rho}$$

the final result is

$$\frac{dR}{R} = \frac{dl}{l} - \frac{dS}{S} + \frac{d\rho}{\rho}$$

where

- dS/S is the variation of area due to the length variation in orthogonal directions
- the strain is the term $\varepsilon = dl/l$

the variations of **width** w and **height** h depend on the Poisson's ratio according to the following equations:

$$dw = -\nu w \varepsilon$$

$$dh = -\nu h \varepsilon$$

the negative sign indicates that a strain ε leading to a positive variation of the length produces a negative variation of w and h

considering a stretching $\varepsilon > 0$ of the transducer, width and height become

$$w - dw \qquad\qquad h - dh$$

due to the stretch, the cross-sectional area is

$$\overline{S} = (w - dw)(h - dh) = wh - 2\nu wh\varepsilon + \nu^2 wh\varepsilon^2$$

assuming that the stretch ε is small, the higher order term $\nu^2 wh\varepsilon^2$ can be neglected:

$$dS = \overline{S} - S = -2\nu wh\varepsilon$$

Strain and resistance

the relationship between resistance and deformation can be written as

$$\frac{dR}{R} = \frac{d\rho}{\rho} + (1 + 2\nu)\varepsilon$$

usually, the **gauge factor** is defined as

$$G = \frac{dR/R}{\varepsilon}$$

that can be rewritten as

$$G = \frac{d\rho/\rho}{\varepsilon} + (1 + 2\nu)$$

the gauge factor is written in the following form

$$G = \frac{d\rho/\rho}{\varepsilon} + (1 + 2\nu)$$

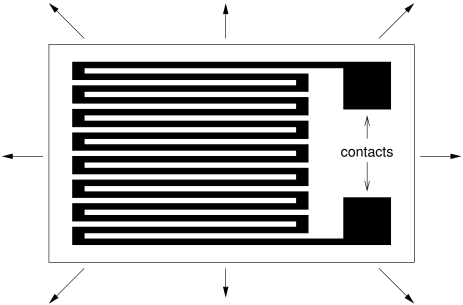
the formula puts the emphasis on two factors:

- a piezoresistive effect due to $(d\rho/\rho)/\varepsilon$
- a geometric effect due to $1 + 2\nu$

by carefully selecting the building material, it is possible to let one effect to dominate the other

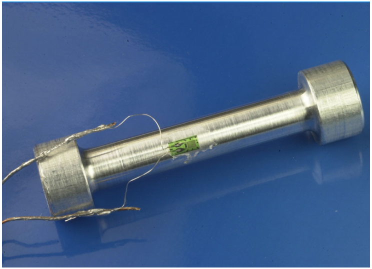
Metal foil strain gauge

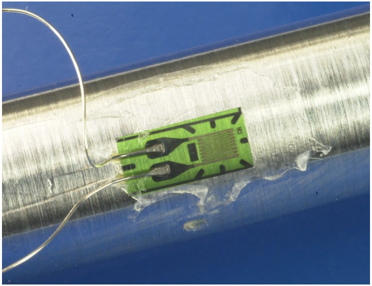
a common type of strain gauge uses a metal foil:



in this kind of sensor,
the piezoresistive effect is dominant

a metal foil strain gauge is used to measure the deformation of a metal bar





source: http://www.doitpoms.ac.uk/

position sensors based on capacities leverage the relationship between **capacitance** and **shape**, **dimension** and **permittivity** of the material

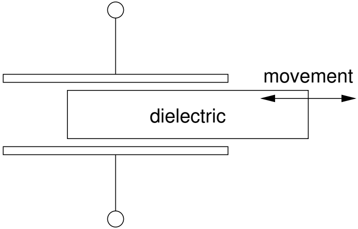
the capacitance of a parallel plate capacitor is

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

where

- ϵ_0 is the free-space permittivity
- ϵ_r is the relative permittivity due to the dielectric material
- A is the area of the surface of plates
- d is the distance between plates

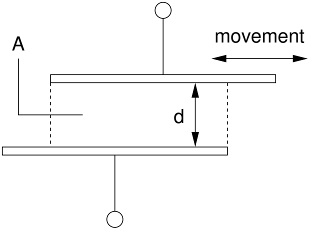
Moving dielectric capacitor



the moving element is the dielectric block that separates the two plates

the changing element is $\epsilon = \epsilon_0 \epsilon_r$

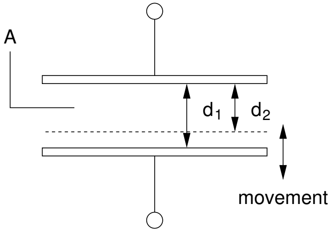
Variable area capacitor



the moving element is one of the two plates that shifts w.r.t. the other one

- the measurement is based on the variation of A
- the distance d is kept constant

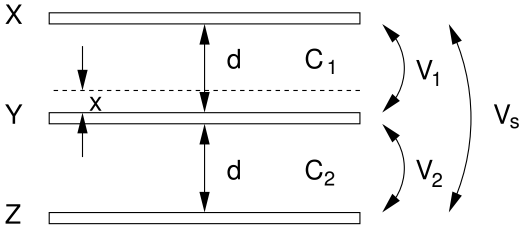
Variable distance



the moving element is one of the plates,
that moves closer/farther to the other one

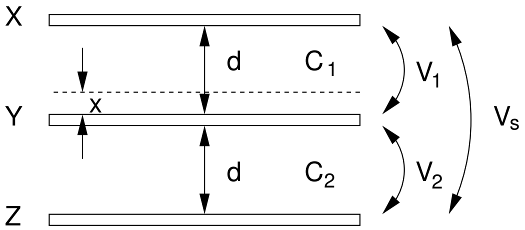
- the changing parameter is d
- the area A is kept constant

Differential capacitive sensor



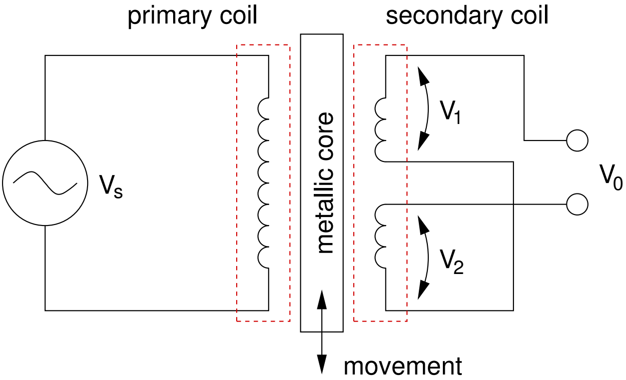
$$C_1 = \frac{\epsilon_0 \epsilon_r A}{d - x} \qquad C_2 = \frac{\epsilon_0 \epsilon_r A}{d + x}$$

Differential capacitive sensor



$$(V_1 - V_2) = V_s \left(\frac{C_2}{C_1 + C_2} - \frac{C_1}{C_1 + C_2} \right) = V_s \frac{x}{d}$$

- the relationship is linear if $|x| < d$
- measurement ranges between 10^{-11} and 10^{-2} meters
- when the capacitance is less than 1 pF, environmental parameters become important, like bonding capacitance, humidity and temperature



$$V_s = \sin(\omega t)$$

$$V_1 = k_1 \sin(\omega t - \phi) \qquad V_2 = k_2 \sin(\omega t - \phi)$$

$$V_1 = k_1 \sin(\omega t - \phi)$$
$$V_2 = k_2 \sin(\omega t - \phi)$$

- the values of k_1 and k_2 depend from the coupling between primary and secondary coils
- it holds $V_0 = V_1 - V_2$
- in the central position it holds $k_1 = k_2 = k$ and $V_1 = V_2$, thus $V_0 = 0$

- when the core moves of x the coupling changes, becoming $k_1 = k_a$ and $k_2 = k_b$

$$V_0 = (k_a - k_b)\sin(\omega t - \phi)$$

- while if the core moves of $-x$ the coupling changes becoming $k_1 = k_b$ and $k_2 = k_a$

$$V_0 = (k_b - k_a)\sin(\omega t - \phi)$$

- therefore

$$V_0 = (k_a - k_b)\sin(\omega t - (\pi - \phi))$$

Inductive technology: considerations

- can measure displacements from 100μm to 100 mm
- in practice, there is no friction
- lifetime up to 200 years

objective

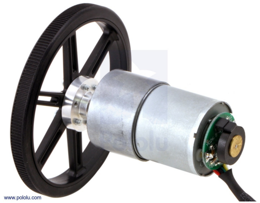
measure the angular displacement of a body,
which can usually rotate along one or more axis

applications:

- **industrial robot motors control**: the angular displacement of the motor shaft determine the positioning of the actuator
- **motion speed of a mobile robot**: from the angular position of the motor shaft the angular speed can be inferred, and the robot motion can be tracked (usually with very low accuracy)

Angular position sensor

the measurement of the angular position
is a key factor for using motors



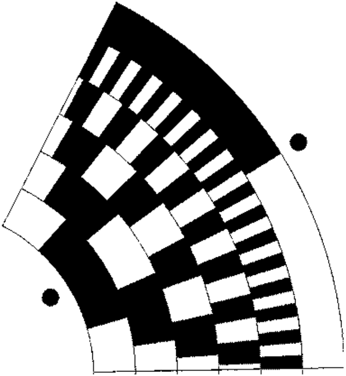
http://www.pololu.com

several technologies can be used:

- resistive (potentiometer)
- inductive (resolver or synchro)
- optical (encoder)

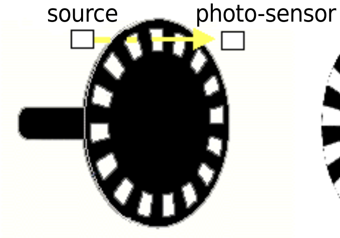
since resistive and inductive technologies have been already shown, we concentrate on the optical technology example of disc composing the absolute encoder

detail of a disc composing the absolute encoder

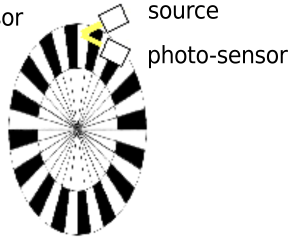


Detection method

two possible manufacturing realizations:

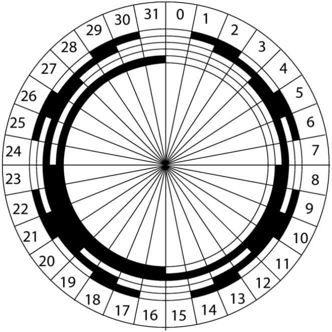


pass-through light



reflected light

The absolute encoder



http://www.ab.com

- an unique binary code is associated to each position of the axis
- it does not require the calibration of the home position
- uses a specific coding to minimize the errors during the reading

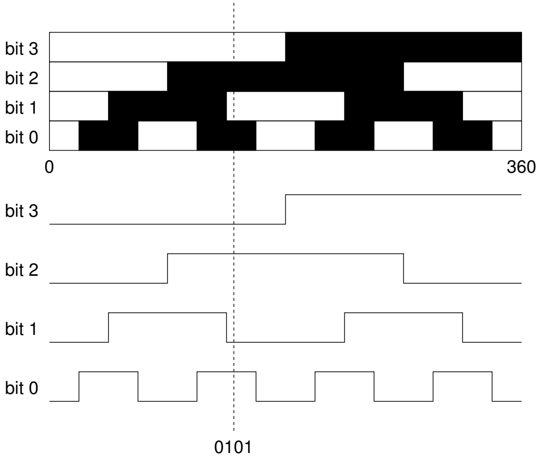
- the resolution is given by the number of concentric circles
- in miniaturized components the number of circles is limited by manufacturing constraints

Absolute Encoder AD 34

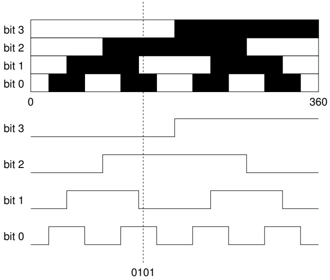
- For brushless servo motors
- Light duty encoder
- Notched shaft 6 mm
- Mounting Depth: 25 mm
- Up to 17 Bit Resolution
- +120°C operating temperature
- 10.000 rpm continous operation
- BiSS or SSI interface
- Sinewave 1 Vpp
- Bandwidth 500kHz

from Hengstler GmBH

Output signals



Gray coding

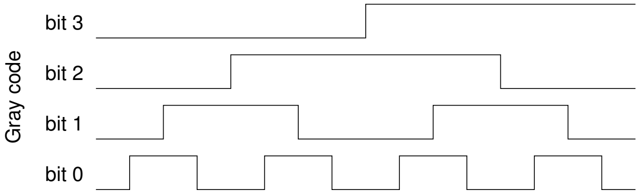


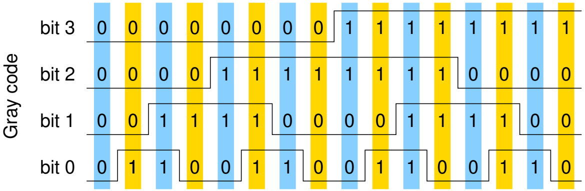
- non-weighted code: no weight is associated to the bits on the basis of their position
- two adjacent code words differ by one symbol only

- minimized reading errors due to keybouncing determined by manufacturing limitations
- speedups the processing

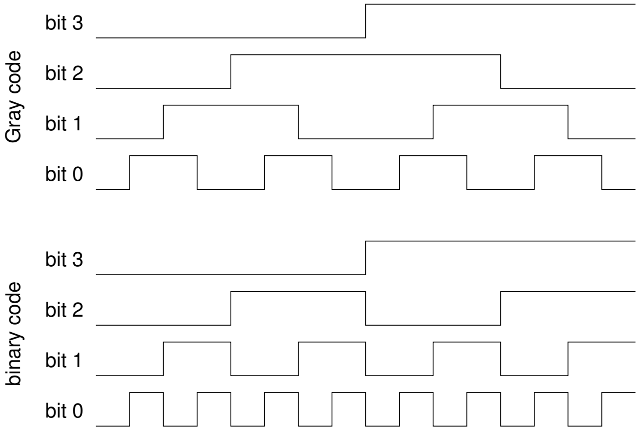
Gray coding

binary codes associated to angular positions

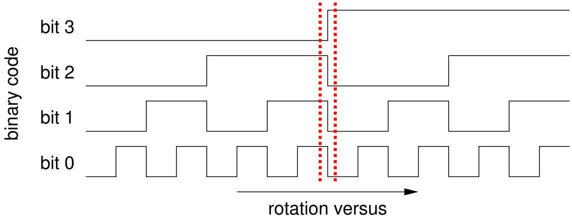




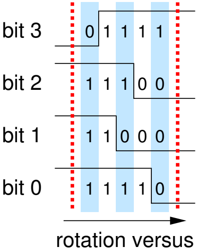
comparison between Gray coding and usual binary coding



example of problem when using the usual binary coding

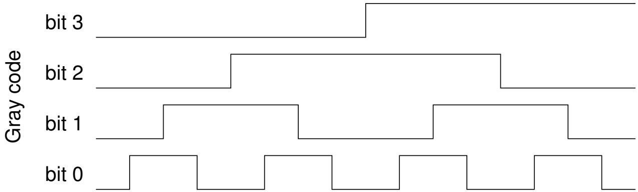


in the transition
0111 $\rightarrow$ 1000
4 bits change
at the same time



due to manufacturing defects, the variation of the 4 bits may not happen at the same time

the actual sequence of change may be, for instance, 0111 $\rightarrow$ 1111 $\rightarrow$ 1101 $\rightarrow$ 1001 $\rightarrow$ 1000 the illustrated problem is solved by the Gray coding

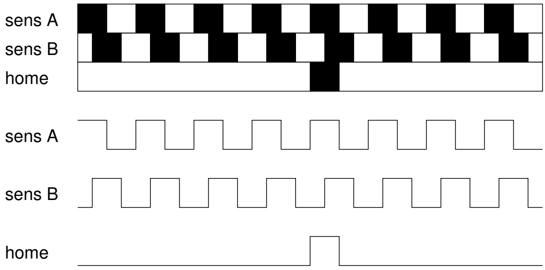


- no transition requires the change of more than 1 bit
- manufacturing defects do not affect the sensor reading

The incremental encoder

- no unique codes are assigned to the shaft position
- allows to track both clockwise and anti-clockwise rotations
- requires the calibration of the home position
- suitable for high rotation speeds

Output signals



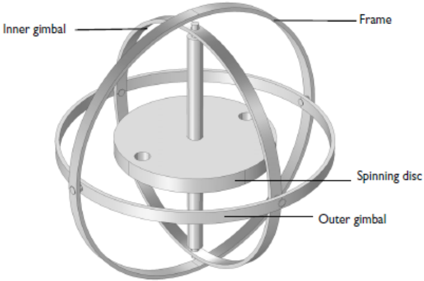
- the home position is needed for the initial calibration
- the current position is tracked through the input variation of two light sensors
- the two signals are phase shifted of 1/4 of period
- it is possible to determine the sense of rotation by observing which front appears first

measures the angular position or speed
of a body rotating on its axis

gyros can use different building technologies:

- mechanical
- optical
- integrated circuits (MEMS)

Mechanical gyro



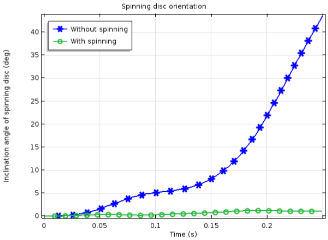
https://www.comsol.com

working based on the law of conservation of angular momentum

- a mass is controlled to rotate at constant speed
- the mass is linked to the chassis by a Cardan joint (universal joint)

Mechanical gyro

- the chassis is integral with the body to monitor
- the rotating mass tends to maintain its rotating axis unchanged even though the chassis is rotated



https://www.comsol.com

the Cardan joints allows the chassis to rotate while the rotating mass keeps its rotating axis fixed in the 3D space

- encoders are mounted on Cardan joints
- the angular position is provided directly by the encoders

MERIDIAN SUBSEA GYROCOMPASS

- Meridian Subsea and Subsea RP
- Maintenance-free DTG element
- Dynamic heading accuracy of ±0.6°
- Static heading accuracy of <0.25°
- <45 minutes settling time
- Start-up power requirement of 1.8A
- Low cost of ownership
- MTBF of 30,000 hours
- Depth rated to 3000m
- Very high turn rate of 200° per second
- Configuration via PC interface S/W
- Optional integral Roll & Pitch module
- Optional internal battery back-up

source: VT Group

Mechanical gyro: example of datasheet

MERIDIAN SUBSEA

TECHNICAL SPECIFICATIONS	
Settle point	0.25° secant latitude
Static accuracy	<0.25° RMS secant latitude
Dynamic accuracy	<0.6° secant latitude (Scorsby and Intercardinal motion tests)
Follow up speed	200°/sec
Settling time	<45 minutes, to within 0.7°
Latitude input	Automatic – via RS232 or RS422, NMEA 0183 from SDC software
Speed input	Automatic – via RS232 or RS422, NMEA 0183 from SDC software
Latitude compensation	80N to 80S
Speed compensation	0 – 20 knots
Operating temperature	0°C to +45°C (to ISO 8728)–15°C to +55°C (with reduced accuracy)
Storage temperature	–25°C to +80°C
Gimbal limits	±45° pitch and roll
Shock survival	10g
Mean time before failure	30,000 hours
Service interval	No schedule maintenance, calibration recommended every 2 years
Input voltage	24VDC (18-36 VDC)
Start-up current	1.8A
Dimensions	516 _ Ø215mm {20.31 _ Ø8.46 inch}
Weight	28.6kg {74.96 pounds} in air
Depth rating	3000m
Accessories included	Operators handbook, transit case, spare connectors
Standards	IMO A 424 (X1), IMO A 821 (1bv9), BS EN 60945, BS EN ISO 8728 1994, BS 6217 1981, CE Marking, Electromagnetic Compatibility (EMC) Directive and the Marine Equipment Directive 96/98/EC
Warranty	12 months international warranty including parts and labour
OPTIONS	
Roll & Pitch Module	Accuracy 0.1° at +/- 37°, update rate 50 Hz, TSS1 or HHRP output formats
Battery Back-up Module	Internal auto-recharging batteries giving up to 1 minute back-up power supply
	Due to continuous development, specifications may vary from those listed above.

source: VT Group

Mechanical gyros: pros and cons

cons

- it has moving mechanical parts; the friction produces errors in the measure and wearing of components
- special bearing and lubricating are needed to reduce the friction, leading to bigger size, weight and cost
- the sensor must warm-up to let all the parts to reach the dilation required to work in proper conditions

pros

- the measure is stable: the rotating mass is able to keep aligned with the global reference system better than every other gyro

open/close an electric circuit depending on the proximity of an object
some sensors may return the distance of the object

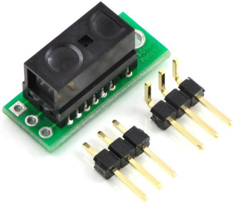
adopted technologies:

- infrared
- LASER
- capacitive
- inductive
- acoustic signals (ultrasound sensors)

applications:

- **indutrial processes**, to detect the presence or the position of machine parts or objects
- **mobile robotics**: obstacle avoidance, localization and mapping
- **security**: to detect open/close doors or windows

Infrared proximity sensors



http://www.karlssonrobotics.com

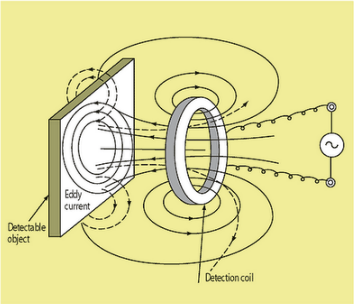
- a photo-emitter generates an infrared light beam
- a photo-sensor detects the reflection of the impulse

possible disturbs caused by environmental illumination

to overcome the problem:

- some sensors modulate the signal to distinguish it from the environmental noise
- other sensors can infer the obstacle distance from the intensity of the perceived light

Inductive proximity sensor



- an oscillator produces a variable magnetic field, feeding a coil
- the magnetic field induces a current in a close object
- the current causes a variation of the magnetic field
- the variation changes the amount of current in the coil

Inductive proximity sensor

- working principle similar to the metal detector
- an electronic circuit measures the variation of current in the coil

> inductive sensors can only detect
> objects made of metal

Inductive proximity sensor: application

Machine position verification: the proximity sensor is used to detect the position of a part on the machine or equipment itself.

Source: https://automation-insights.blog/2010/08/31/3-common-applications-for-discrete-output-inductive-proximity-sensors/

Characteristics of the E2E-X2D1-N sensor

www.omegamation.com

range	[2..20] mm
max pressure	83 bar
power supply	[10..30] volt
output	[3..30] mA
temperature	[−25..70] Celsius degrees
cost	around 80$

Capacitive proximity sensors

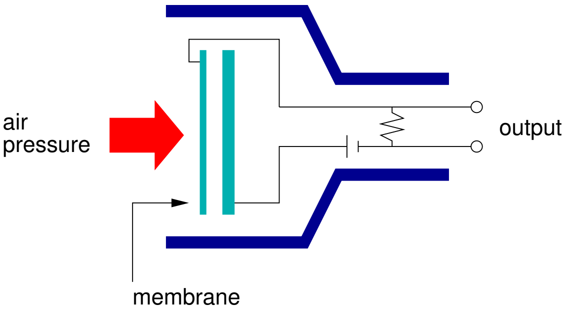
- detect the variation of capacitance produced by an object
- a radio-frequency oscillator is connected to a metal plate
- when the plate gets close to an object the oscillation frequency changes, detecting the object

> capacitive sensors can only detect objects made of metal

Ultrasonic sensor

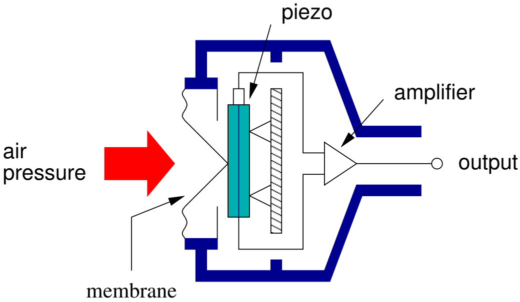
- very useful for range detection in mobile robotics, together with laser scanner and cameras
- the working principle is the same as SONARs (SOund Navigation And Ranging)
- active sonars emit the acoustic signal and detect the reflected one
- working range up to 10 meters
- immune to electromagnetic noise
- can detect objects made of any material
- echo may not be detected in case of small objects of unfavourable orientations

scheme of a condenser microphone



Piezo microphone

scheme of a piezo microphone



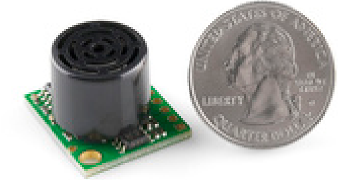
- the piezo effect is reversible
- the same component can be used to generate the signal

Ultrasonic proximity sensor: characteristics

parameter	value	unity
voltage	5	[V]
current	30–50	[mA]
frequency	40	[Khz]
range	30–3000	[mm]
sensitivity	3 at dist. $>$ 200	[cm]
input trigger	10	[μs]
size	43 $\times$ 20 $\times$ 17 (H)	[mm]

the sensor generates an impulse having duration proportional to the distance

Ultrasonic sensor: characteristics



parameter	value	unity
voltage	[2.5..5.5]	[V]
current	2	[mA]
frequency	42	[Khz]
range	[0..6.45]	[m]
PWM sensitivity	147	[μs/inch]
analog sensitivity	10	[mV/inch]
cost	25	dollars

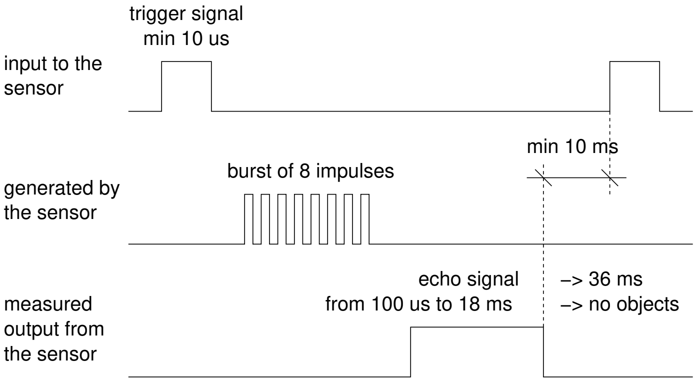
source: Maxbotix

example of ultrasonic sensor

- one ultrasonic component emits the signal, the other receives the reflected one
- the two components are, in practice, piezo microphones made by ceramic materials

Input/output signals

example of signal timing involved
in the behavior of an ultrasonic sensor

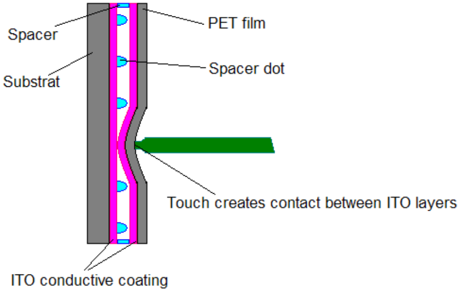


more and more adopted as a input device for electronic appliances (PC, palm, etc.)

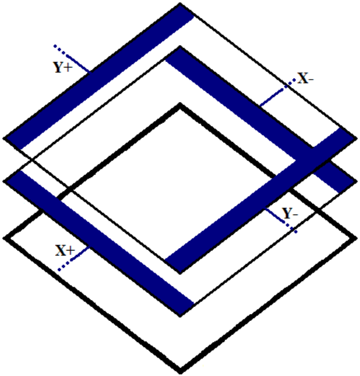
several technologies are used:

- resistive (4 or 5 wires)
- capacitive
- acoustic
- optical (vision)

Touchscreen

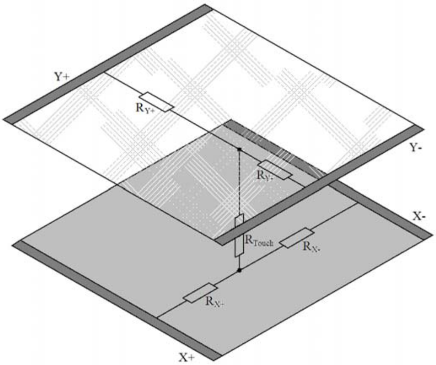


- the external surface is in PET and a rigid glass or acrylic
- conductive layers are made by indium oxide (ITO - Indium Tin Oxide)
- conductive layers are interleaved by an empty space

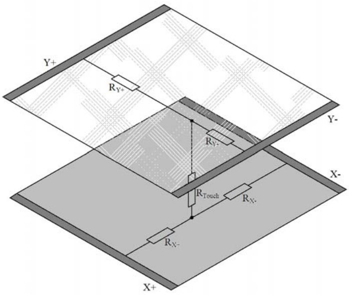


- two contacts, called busbars, on each ITO layer
- busbar on different layers are orthogonal
- 4 wires connect the busbar: left X+, right X-, high Y+, low Y-

4 wires touchscreen: electrical circuit



- the pressure produces a contact between the two conductive layers
- the generated electrical circuit is depicted in the figure
- notice the R_{touch} resistance



	X+	X-	Y+	Y-
standby	GND	Hi-Z	Hi-Z	Pull-up
X axis	GND	Vcc	Hi-Z	Hi-Z / ADC
Y axis	Hi-Z	Hi-Z / ADC	GND	Vcc

track the absolute position of a point
on the Earth surface

applications:

- navigation systems
- monitoring/tracking of motion (outdoor)
- distributed clock synchronization

GPS: Global Position System

- positioning system based on satellites
- global and continuous coverage
- the positioning system is operated by the Department of Defense (DoD)
- 24 satellites at 20200 Km altitude
- orbit data are updated when satellites transit over the US
- Doppler signals are used to estimate their updated position
- orbit data are retransmitted to the satellite
- each satellite periodically sends its data to the receiver
- sent data include the position (x, y, z) and the sending time t

Alternative to GPS

- **GLONASS**: Globalnaya Navigatsionnaya Sputnikovaya Sistema (Russia)
- DORIS: Doppler Orbitography and Radiopositioning Integrated by Satellite (France)
- BeiDou/COMPASS (China)
- Galileo (EU)
- IRNSS: Indian Regional Navigation Satellite System (India)
- QZSS: Quasi-Zenith Satellite System (Japan); integration to GPS

GPS

- the GPS requires 3 satellites to calculate its 2D position on the Earth surface
- 4 satellites are required to calculate the altitude (3D positioning)
- each satellite sends two types of signals at 1.5 and 1.2 GHz containing its position, and the almanac
- the information is sent at 50 bit/sec
- the almanac is used for the calibration, when the receiver is switched on after long time
- the transmission of the almanac takes 12.5 minutes

GPS sensor module

small and lightweight sensor module

when the GPS signal arrives to the receiver, the following information are known:

- the transmission time t_s (sent with the signal)
- the arrival time t_r
- the position (x, y, z) of the satellite

the distance satellite-receiver D can be calculated as

$$D = c(t_r - t_s)$$

where c is the speed of light

once the distance from 3/4 satellites is known, together with the position of satellites, the technique of trilateration can be applied to calculate the receiver's position

source: http://www.mio.com/

due to the effect of relativity, calculations are much more complex than those here presented

Triangulation, trilateration, multilateration

> there are different methods to calculate an unknown position starting from the positions of known reference points

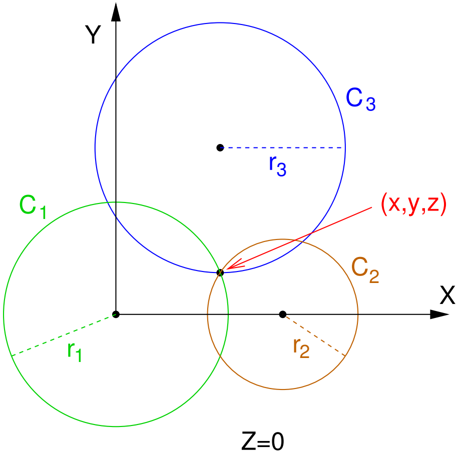
the following 3 methods are based on different parameters:

- **triangulation**: uses the distance from known locations and the angles of lines connecting the receiver and the known locations
- **trilateration**: only the distance from known locations is used
- **multilateration**: based on the TDOA (Time Difference Of Arrival), i.e., the time difference of the signal that is received by 3 or more receivers at known locations

the 3 methods require a travelling signal used to detect the distance between emitter and receiver

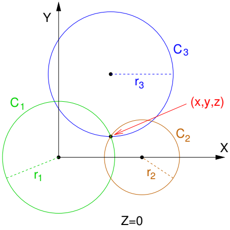
- in trilateration, the signals are emitted by the reference points at known positions and sensed by the point at unknown position
- in multilateration, the signal is emitted by the point at the unknown location and it is received by the reference points

the transmitted signal is usually a radio signal or a sound (or ultra-sound) signal



there are three circles those known centers are the source of signals to consider

Trilateration



using an adequate coordination adaptation, it can be obtained:

- C_1 has radius r_1 and center in $(0, 0, 0)$
- C_2 has radius r_2 and center in $(x_2, 0, 0)$
- C_3 has radius r_3 and center in $(x_3, y_3, 0)$

notice that $z = 0$ for every circle

the equations describing the circles are:

$$r_1^2 = x^2 + y^2 + z^2 \tag{1}$$
$$r_2^2 = (x - x_2)^2 + y^2 + z^2 \tag{2}$$
$$r_3^2 = (x - x_3)^2 + (y - y_3)^2 + z^2 \tag{3}$$

substracting (2) from (1):

$$r_1^2 - r_2^2 = x^2 + y^2 + z^2 - (x - x_2)^2 - y^2 - z^2$$

$$r_1^2 - r_2^2 = 2xx_2 - x_2^2$$

the result is

$$x = \frac{r_1^2 - r_2^2 + x_2^2}{2x_2} \tag{4}$$

then, the substitution of (4) into (1) leads to

$$y^2 + z^2 = r_1^2 - \frac{(r_1^2 - r_2^2 + x_2^2)^2}{4x_2^2} \tag{5}$$

z^2 results from (3) and (5) by setting equals

$$r_1^2 - \frac{(r_1^2 - r_2^2 + x_2^2)^2}{4x_2^2} - y^2 = r_3^2 - (x - x_3)^2 - (y^2 - 2y_3y + y_3^2)$$

by setting (5) equal to (3), and with adequate simplifications, the results is

$$y = -\frac{r_3^2 - r_1^2 - (x - x_3)^2 - y_3^2}{2y_3} - \frac{(r_1^2 - r_2^2 + x_2^2)^2}{8x_2^2 y_3} \tag{6}$$

finally, known x and y, from (1):

$$z = \sqrt{r_1^2 - x^2 - y^2}$$

Multilateration

- based on the Time Difference Of Arrival (TDOA), i.e., the difference of the arrival time of signals to the set of receivers having known locations
- to track the position of a point in the space 4 sensors are required
- the departure time of signals is not required
- clocks of the receiving points need to be synchronized
- it suffices to known the difference of the arrival times registered by the sensors

the localization system has the following requirements:

- an emitter having unknown position (x, y, z)
- 4 sensors S_i, with $i = [1, 2, 3, 4]$
- the i-th sensor has known coordinate $P_i = (x_i, y_i, z_i)$
- the signal has known speed v (often $v = c$, whene c is the speed of light)

the signal travelling time can be calculated on the basis of the signal speed and the unknown distance between the emitter and the receivers:

$$T_1 = \frac{1}{v}\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

$$T_2 = \frac{1}{v}\sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}$$

$$T_3 = \frac{1}{v}\sqrt{(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2}$$

$$T_4 = \frac{1}{v}\sqrt{(x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2}$$

- the procedure calculates the TDOA τ_i between pairs of sensors
- TDOAs are referred to one reference sensor (e.g., T_4)
- the reference sensor is supposed to be located at the origin of the reference system

above observations allow to state that

$$T_4 = \frac{1}{v}\sqrt{x^2 + y^2 + z^2}$$

the equations to considers become

$$\tau_1 = T_1 - T_4 = \frac{1}{v}\left(\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} - \sqrt{x^2+y^2+z^2}\right)$$

$$\tau_2 = T_2 - T_4 = \frac{1}{v}\left(\sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} - \sqrt{x^2+y^2+z^2}\right)$$

$$\tau_3 = T_3 - T_4 = \frac{1}{v}\left(\sqrt{(x-x_3)^2+(y-y_3)^2+(z-z_3)^2} - \sqrt{x^2+y^2+z^2}\right)$$

all parameters, including τ_1, τ_2, τ_3, are known

- the solution of the system of equations allows to obtain the position of the emitter
- the solution is pretty complex
- however, it can be be solved in closed form
- therefore, the exact solution can be obtained

Multilateration: accuracy

several factors affect the accuracy of the localization based on multilateration:

- the relative positions of receivers
- the accuracy of the time-stamping of received signal
- the accuracy of time synchronization between receivers
- the bandwidth of the signal
- the accuracy of localization of receivers